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Compression fracture statistics of compacted cement cylinders

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A statistical analysis of the compression fracture stress of compacted cement paste cylinders was made by comparison of normal, logarithmic normal and Weibull distributions. The influence of specimen size on the fracture stress was determined and compared with predictions based on crack propagation results. The results of the analysis show that the Weibull modified statistics fit in very well with the behaviour observed.

1. Introduction

In uniaxial compression de Jayatilaka and Trustrum [1] predicted that the failure stress of brittle materials should follow closely a normal distribution with mean values independent of the volume, and variance inversely proportional to the volume. In the case of uniaxial tensile stress, propagation of a single crack leads to total failure, which is, in general, not true during uniaxial compression. So, if at a constant tensile stress field σ, the volume V is arbitrarily decomposed into V_1 and V_2, the following equation must hold:

$$\tilde{F}(V_1 + V_2, \sigma) = \tilde{F}(V_1, \sigma)\tilde{F}(V_2, \sigma) \qquad (1)$$

$$V_1 \cap V_2 = \phi, V_1 \cup V_2 = V$$

$\tilde{F}(V_1 + V_2, \sigma)$ is the cumulative probability of non-failure in the volume $V = V_1 + V_2$, where V_1 and V_2 have no common points, which is the meaning of $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$. $\tilde{F}(V_i, \sigma)$ is the cumulative probability of non-failure in V_i. The functional Equation 1 can be easily transformed to the following differential equation:

$$\frac{\mathrm{d}\tilde{F}}{\tilde{F}} = \text{constant} \times \mathrm{d}V \qquad (2)$$

with the boundary conditions:

$$\tilde{F}(0, \sigma) = 1, \quad \tilde{F}(\infty, \sigma) = 0 \qquad (3)$$

The first condition can be deduced from Equation 1 and the second means that if a body is sufficiently large, it must break at a finite tension $\sigma \geqslant \sigma_1$. According to this, the constant of Equation 2 must be a negative function of σ. The function $\phi(\sigma)$ is crescent with σ. Taking this into account, Equation 2 becomes:

$$\frac{\mathrm{d}\tilde{F}}{\tilde{F}} = -\frac{\mathrm{d}V}{V_0} \times \phi(\sigma) \qquad (4)$$

where V_0 is the unit volume and $\phi(\sigma)$ can be defined as the specific risk of failure. Integration of Equation 4, with the boundary conditions given in Equation 3, gives

$$\tilde{F}(V, \sigma) = \exp\left[-\frac{V}{V_0}\phi(\sigma)\right] \qquad (5)$$

The final formula for the cumulative probability of failure $F(V, \sigma) = 1 - \tilde{F}(V, \sigma)$ is:

$$F(V, \sigma) = 1 - \exp\left[-\frac{V}{V_0}\phi(\sigma)\right] \qquad (6)$$

which is valid for uniaxial constant tensile stress. In the case of crack propagation instead of failure Equation 6 is valid for both tensile and compression stress. Equation 1 can be easily generalized to

$$\tilde{F}\left[\sum_{i=1}^{N}(V_i, \sigma_i)\right] = \prod_{i=1}^{N}\tilde{F}(V_i, \sigma_i) =$$

$$= \exp\left[-\frac{1}{V_0}\sum V_i\phi(\sigma_i)\right] \qquad (7)$$

In the limit, when $V_i \to 0$, $N \to \infty$, the celebrated

Weibull [2, 3] formula for variable uniaxial stress is obtained:

$$F(\sigma) = 1 - \exp\left\{-\frac{1}{V_0}\int_V \phi[\sigma(r)]\,\mathrm{d}V\right\} \quad (8)$$

where $\sigma(r)$ is the uniaxial stress field.

The objective of this work is to confirm Equation 8, for a cement paste cylinder tested in compression, with the de Jayatilaka and Trustrum prediction.

2. Experimental methods

The samples were prepared with commercial Portland Cement without additives having the following composition: 20.42% S; 7.16% A; 2.76% F; 61.40% C and 1.72% M (all wt%) specific surface of $305\,\mathrm{m^2\,kg^{-1}}$, and a retention of 13.6% in a moist $45\,\mu$m sieve. The cement was mixed with 3 wt% water, and was placed in cylindrical steel moulds with a radius to height ratio, $R/H = 0.25$, and R values of: 0.0095, 0.011 and 0.013 m. The cement placed in the moulds was compressed at a pressure of 27.5 MPa ($280\,\mathrm{kgf\,cm^{-2}}$). This pressure was sufficient to produce good cohesion to avoid handling problems. The specimens obtained were a batch of 575 with $R = 0.0095$ m; another batch of 30 with $R = 0.0095$ m, 30 with $R = 0.011$ m and 30 with $R = 0.013$ m. To improve further the overall resistance, the specimens were kept in a humid chamber for 24 h. In order to complete hydration ($\simeq 30\%$ water), the samples were submerged in water to saturation and again placed in the humid chamber for six days in order to finish the curing process. They were then heat treated at 100° C for 24 h to stop hydration, and stored in a desiccator until they were tested in an Amsler manual machine by compression.

The compression stress of failure, for the first 575 samples ($R = 0.0095$ m), was ordered in intervals of 0.49 MPa ($5\,\mathrm{kg\,cm^{-2}}$), and for each interval the cumulative probability of failure $F(\sigma)$ was computed.

In order to make a statistical comparison, the Kolmogorov–Smirnov limits [4] for 95% confidence, and the χ^2 test, were performed for different distributions of the 575-samples set.

We took into account the fact that the data have been used to estimate the model parameters by simply reducing the number of degrees of freedom of the distribution of the statistics by one, for each parameter estimated from the data [4]. Rearranging the lower populations' intervals, the same number of degrees of freedom was obtained, and was 27.

For the normal distribution the following values are obtained:

$$F(\sigma) = \frac{1}{S_\sigma(2\pi)^{1/2}}\int_{-\infty}^{\sigma}\exp\left[\frac{(\sigma-\bar{\sigma})^2}{2S_\sigma^2}\right]\mathrm{d}\sigma$$

$$\bar{\sigma} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i = 131.4\,\mathrm{MPa}\ (1340\,\mathrm{kgf\,cm^{-2}})$$

$$S_\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(\sigma_i-\bar{\sigma})^2 \quad (9)$$

$$S_\sigma = 11.60\,\mathrm{MPa}\ (118\,\mathrm{kgf\,cm^{-2}}),$$

$$\chi^2 = 30.8 < 40.1$$

For this distribution and the corresponding number of degrees of freedom, and 95% confidence, $\chi^2 = 40.1$. Fig. 1 shows that the Kolmogorov–Smirnov test indicates a reasonable fit.

For the normal logarithmic distribution, the Kolmogorov–Smirnov test indicated a better fit to the data (Fig. 2), and the calculations for the χ^2 test are:

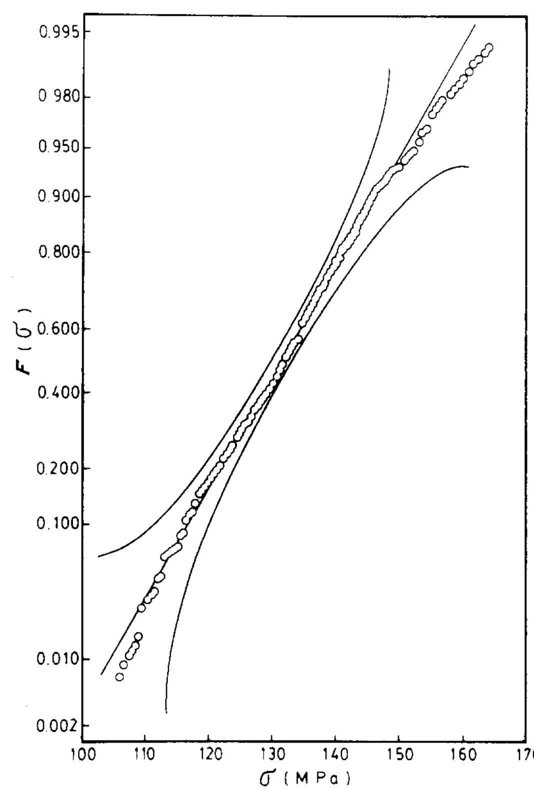

Figure 1 Kolmogorov–Smirnov test for 95% confidence in a normal distribution for compression.

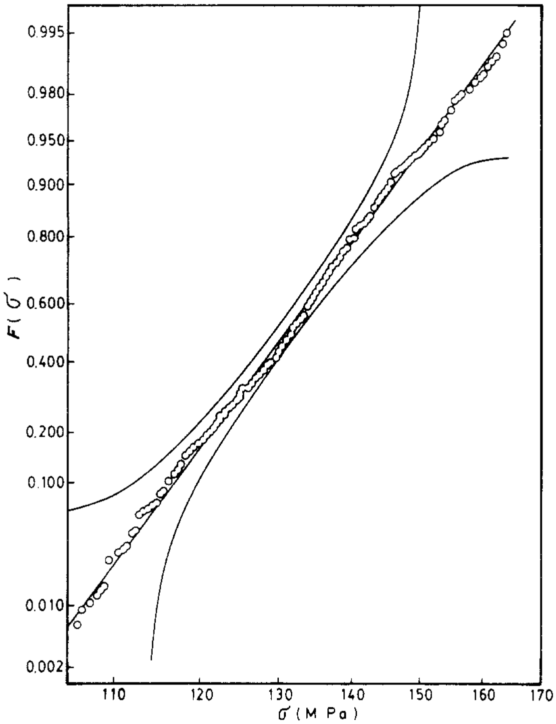

Figure 2 Kolmogorov–Smirnov test for 95% confidence in a normal logarithmic distribution. Same test as in Fig. 1.

$$F(\sigma) = \frac{1}{S_y(2\pi)^{1/2}}\int_{-\infty}^{y}\exp\left[-\frac{(y-\bar{y})^2}{2S_\sigma^2}\right]\mathrm{d}y$$

$$y = \ln\sigma, \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i = 4.87$$

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y-\bar{y})^2,\ S_y = 0.088$$

$$\chi^2 = 27.7 < 40.1 \tag{10}$$

In the case of the Weibull distribution, for uniaxial compressive stress taking the Weibull function we obtain

$$\phi(\sigma) = \left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m,\ \sigma \geqq \sigma_1$$

$$\phi(\sigma) = 0, \qquad \sigma < \sigma_1 \tag{11}$$

where σ_0, m and σ_1 are the Weibull parameters, assuming that the stress field $\sigma(x, y, z) \simeq \sigma = P/\pi R^2$, where P is the applied compression force. In accordance with Equations 8 and 11 and after some rearrangement:

$$\ln\left\{\ln\left[\frac{1}{1-F(\sigma)}\right]\right\} = m\ln(\sigma-\sigma_1) + \ln\left(\frac{V}{V_0\sigma_0^m}\right) \tag{12}$$

is obtained. Fig. 3 shows a plot of $\ln(\ln\{1/[1-F(\sigma)]\})$ as a function of $\ln(\sigma-\sigma_1)$. With a σ_1 that maximizes the correlation coefficient for a straight line, the following constants are obtained:

$$\sigma_1 = 97.1\,\mathrm{MPa}\ (990\,\mathrm{kgf\,cm^{-2}})$$

$$m = 3.12$$

$$\sigma_0 = 80.8\,\mathrm{MPa}\ (824\,\mathrm{kgf\,cm^{-2}}) \tag{13}$$

$$\chi^2 = 31.2 < 40.1$$

In order to investigate the dependence of $F(\sigma)$ on specimen size a Weibull plot was drawn (Fig. 4) with the 30 samples each of $R = 0.0095$, 0.011 and 0.013 m. Using different volumes and Equation 12 we obtained

$$\ln\left\{\ln\left[\frac{1}{1-F_{V_2}(\sigma)}\right]\right\} - \ln\left\{\ln\left[\frac{1}{1-F_{V_1}(\sigma)}\right]\right\} = \ln\left(\frac{V_2}{V_1}\right) \tag{14}$$

In accordance with Equation 14 the Weibull plot for a batch of samples with $V_2 > V_1$ is the plot of V_1 plus $\ln(V_2/V_1)$. Fig. 4 shows what can be expected for V_1 ($R = 0.0095$ m) and V_2 ($R = 0.011$ and 0.013 m). As is evident no volume influence can be detected.

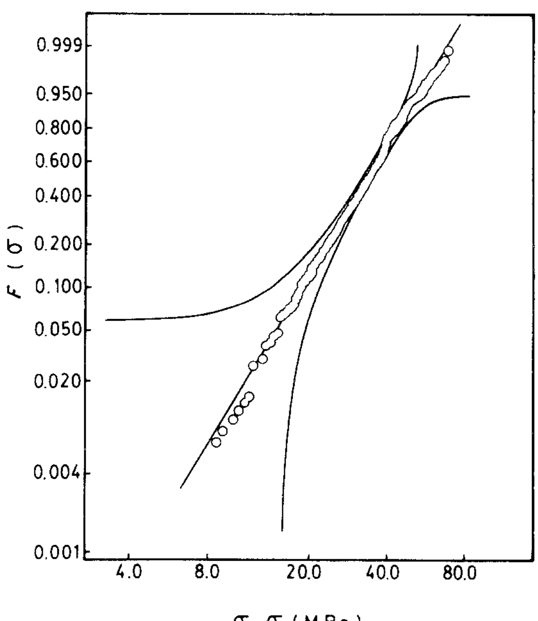

Figure 3 Kolmogorov–Smirnov test for 95% confidence in a Weibull distribution. Same test as in Figs. 1 and 2.

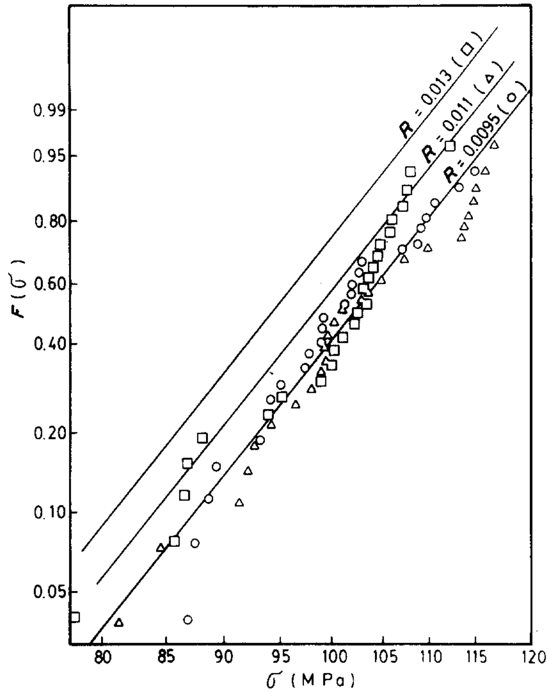

Figure 4 The non-influence of volume in the cumulative probability of failure.

3. Discussion

The normal distribution is better than the Weibull, as is de Jayatilaka's first prediction [1]. But the logarithmic normal is better than the normal, and the χ^2 tests do not reject any of the three. The second de Jayatilaka prediction, the non-dependence of size, is supported too. But in a recent work [5] the size influence in traction was investigated and negative conclusions were obtained. This is not in agreement with de Jayatilaka's conclusions because if the Weibull distribution is not good for compression, it must be good for tensile stress. So a change in the Weibull distribution is proposed and after [5] the differential Equation 4 changes to:

$$\frac{d\tilde{F}}{\tilde{F}} = -\frac{dV}{V}\phi(\sigma) \qquad (15)$$

which leads to:

$$F(\sigma) = 1 - \exp\left[-\int_{V_m}^{V} \phi(\sigma)\frac{dV}{V}\right] \qquad (16)$$

where V_m is the threshold volume for which failure occurs. The integration of Equation 16 is very difficult because $\sigma(r)$ is unknown, but if as a first approximation, σ can be considered as constant, then Equation 16 gives

$$F(\sigma) = 1 - \left(\frac{V}{V_m}\right)^{-\phi(\sigma)} \qquad (17)$$

This equation is, in a Weibull plot, a straight line, as Equation 12, but predicts that when $V_m/V \to 0$, $F(\sigma)$ is independent of V. Equation 17 gives

$$\ln\left\{\ln\left[\frac{1}{1-F(\sigma)}\right]\right\} = \ln\phi(\sigma) + \ln\left[\ln\left(\frac{V}{V_m}\right)\right] \qquad (18)$$

Taking into account the two different volumes in Equation 18 we obtain

$$\ln\left\{\ln\left[\frac{1}{1-F_{V_1}(\sigma)}\right]\right\} - \ln\left\{\ln\left[\frac{1}{1-F_{V_2}(\sigma)}\right]\right\} = \ln\left(\frac{\ln V_2 - \ln V_m}{\ln V_1 - \ln V_m}\right) \to 0 \qquad (19)$$

$$V_m \to 0$$

Using regression analysis on the data plotted in Fig. 4, and from Equation 19 we obtain

$$\ln\left(\frac{\ln V_2 - \ln V_m}{\ln V_1 - \ln V_m}\right) < 0.25 \qquad (20)$$

and in consequence $V_m < 3.9 \times 10^{-7}\,m^3$ since $V_1 = 1.08 \times 10^{-5}\,m^3$, $V_2 = 1.672 \times 10^{-5}\,m^3$ or $2.76 \times 10^{-5}\,m^3$, so that there is no influence of volume, which is also the case with tensile stress. As a consequence of all this the Kittl–Günther modified Weibull statistics appear to be the most appropriate. These statistics are based on the idea that increasing the volume does not imply increasing the size of flaws, and have the advantage that its parameters can be related to the size and distribution of the flaws. This fact has been shown by de Jayatilaka and Trustrum [1].

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References

1. A. de JAYATILAKA and K. TRUSTRUM, *J. Mater. Sci.* **13** (1977) 455.
2. W. WEIBULL, *Ing. Veterkaps Akad. Handl.* **115** (1939) 45.
3. P. KITTL, *Res. Mech.* **1** (1980) 161.
4. J. R. BENJAMIN and C. A. CORNELL, "Probability, Statistics and Decision for Civil Engineers" (McGraw-Hill Book Company, New York, 1970).
5. P. KITTL and O. GÜNTHER, *Res. Mech. Lett.* **1** (1981) 145.

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